1101
JAN 08

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2008
Mathematics 1101
Monday 7 January 2008 11.30 – 1.30 or 1.15 – 3.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

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1. (a) State what it means for a real sequence to converge.

   (b) State the Least Upper Bound principle (continuum property)

   (c) Prove that every increasing sequence that is bounded above converges.

   (d) State the Bolzano-Weierstrass Theorem.

2. (a) State the definition of $\lim_{x \to \xi} f(x) = l$.

   (b) Let f be continuous on the compact interval $[a, b]$. Prove that f is bounded on $[a, b]$.

   (c) Suppose that $f(y) \to l$ as $y \to \eta$ and that $g(x) \to \eta$, as $x \to \xi$. Also assume that f is continuous at η, i.e. $l = f(\eta)$. Show that

$$f(g(x)) \to l \quad \text{as } x \to \xi.$$

3. (a) Define what it means for a sequence to be Cauchy.

   (b) State the General principle of convergence.

   (c) Prove that the sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is increasing, while the sequence

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1}$$

is decreasing. Show that $x_n < y_n$. Deduce that x_n is bounded above, while y_n is bounded below. Conclude that they have the same limit.

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4. (a) Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Show that $\lim a_n = 0$.

(b) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$.

(c) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{3n}{2n^3 - n}, \quad \sum_{n=1}^{\infty} 4^n \left(\frac{n}{n+1}\right)^{n^2}.$$

5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} (x-1)^2 & (x \leq 1) \\ x^{-1} - 1 & (x > 1). \end{cases}$$

Prove carefully (using ϵ and δ) that $f(x)$ is continuous at $x = 1$.

(b) State and prove the Intermediate Value Theorem.

(c) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$.

Prove that for some $\xi \in [0,1]$ we have $f(\xi) = \xi$.

6. Consider the sequence defined by

$$x_1 = 1, \quad x_{n+1} = \frac{1}{1 + x_n}, \quad n = 1, 2, \ldots.$$

Let $l = (\sqrt{5} - 1)/2$, the positive root of the equation $x^2 + x - 1 = 0$.

(a) Show inductively that $x_{2n} < l$ and $x_{2n-1} > l$.

(b) Show inductively that the subsequence $\langle x_{2n} \rangle$ is increasing, while the subsequence $\langle x_{2n-1} \rangle$ is decreasing.

(c) Explain why $\lim x_n = l$.

END OF PAPER